JOURNAL OF COMPUTATIONAL PHYSICS 53, 227–239 (1984)

Analytical, Linear Stability Criteria for the Leap-Frog, Dufort–Frankel Method

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Received February 25, 1983; revised July 1, 1983

The criterion of linear numerical stability of the combined leap-frog Dufort–Frankel scheme for advective–diffusive problems in two dimensions is

$$\left(\frac{\kappa_x}{\Delta x^2}+\frac{\kappa_y}{\Delta y^2}\right)\left(\frac{U^2}{\kappa_x}+\frac{V^2}{\kappa_y}\right)\Delta t^2 \leqslant 1,$$

where Δx and Δy are the grid spacings in the x and y directions, U and V the velocity components, κ_x and κ_y the diffusion coefficients, and Δt the time step. Although this stability requirement does not depend explicitly on the magnitude of the diffusivity (only on the ratio of the diffusivity coefficients), the presence of the diffusive terms renders the criterion more severe than the one obtained for purely advective problems $[(|U|/\Delta x+|V|/\Delta y)\,\Delta t \leqslant 1]$. Only in one dimension are both criteria identical. Therefore, at more than one dimension, the unconditionally stable scheme (Dufort–Frankel) combined with the conditionally stable scheme (leap-frog) leads to the more restrictive stability condition. The maximum allowable time step occurs when the grid spacings in both directions have the same ratio as the square roots of the diffusivity coefficients.

A new method is presented for discussing the stability of a finite-difference scheme without actually solving for the modulus of the amplification factor. This method is also extended to more general cases.

1. INTRODUCTION

The criterion of linear numerical stability can be determined by the von Neumann method rather easily for various simple schemes including either advection or diffusion, for instance. As a result, it is well known that the leap-frog scheme for purely advective problems needs to meet a CFL condition, and that the Dufort–Frankel scheme for purely diffusive problems is unconditionally stable. When a combination of such simple schemes is employed in practical usage, the rule of thumb is to meet the requirements of every scheme separately. However, this rule has no mathematical foundation, and one must proceed with caution. The purpose of this paper is to show, as an example, that the combined leap-frog Dufort–Frankel scheme widely used for advective–diffusive problems has in fact a more stringent stability

0021-9991/84 $3.00

criterion than either scheme separately. Although the new stability conditions are not dramatic in general, it may be important in particular situations such as the modelling of geophysical fronts (small grid size in one direction and large velocity in the perpendicular direction).

In his treatise on computational fluid dynamics, Roache [5, p. 61] is incorrect.[1] He states without proof that, since the Dufort–Frankel scheme is unconditionally stable and that it does not affect the numerical stability of the leap-frog scheme at one dimension, it should also be so at two dimensions. Schumann [6] correctly states that the combination of schemes leads to more severe conditions but fails to extract the analytical expression of the appropriate criterion, invoking the excuse of "insurmountable mathematical difficulties." Indeed, the mathematics of this problem are not straightforward, and, in view of this difficulty, the stability studies of the combined schemes [1, 3, 4] have been limited to one-dimensional problems. Because, at one dimension, the Dufort–Frankel scheme does not affect the stability of the leap-frog scheme, the authors of [1, 4] fail to mention any unexpected added constraint that one can encounter at two or more dimensions. The purpose of this paper is to demonstrate (i) that it is possible to determine the proper criterion by analytical methods, and (ii) that this criterion is more severe in the presence of diffusion in two dimensions as it was already anticipated in [6].

A last section generalizes the methodology used here to all cases involving a complex quadratic equation. The general criterion, which is then obtained, is found to be nothing but a particular case of a theorem proposed by Miller [2]. Miller's work demonstrates how a series of criteria (that all roots of a polynomial be inside or on the unit circle) can be obtained in a systematic approach by reducing the polynomial to one of a degree less in a recursive manner. However, Miller did not explicitly formulate the criterion corresponding to a quadratic equation. The last section thus provides this formulation as well as an independent and inductive way to obtain it.

2. Problem and Solution

The Problem

The two-dimensional advective–diffusive problem under consideration is

$$\frac{\partial T}{\partial t} + U\frac{\partial T}{\partial x} + V\frac{\partial T}{\partial y} = \kappa_x \frac{\partial^2 T}{\partial x^2} + \kappa_y \frac{\partial^2 T}{\partial y^2},$$

where T is the scalar transported and diffused, U and V are the uniform velocity components, κ_x and κ_y are the diffusivity coefficients in the x and y directions, respectively. The discretized version of this equation using a leap-frog scheme for the advection terms and a Dufort–Frankel scheme for the diffusion terms can be written as

[1] In the second edition of his book (1976), Roache noted his error.

$$T^{n+1} = T^{n-1} - \frac{U\,\Delta t}{\Delta x}(T_{j+1} - T_{j-1}) - \frac{V\,\Delta t}{\Delta y}(T_{k+1} - T_{k-1})$$

$$+ \frac{2\kappa_x\,\Delta t}{\Delta x^2}(T_{j+1} + T_{j-1} - T^{n+1} - T^{n-1})$$

$$+ \frac{2\kappa_y\,\Delta t}{\Delta y^2}(T_{k+1} + T_{k-1} - T^{n+1} - T^{n-1}),$$

where $T^n_{jk} = T(x_j, y_k, t_n)$. The obvious subscripts have been deleted for clarity. Since this linear equation has a solution of the form $T^n_{jk} = T_0 G^n \exp[(jm\,\Delta x + kl\,\Delta y)]$, one obtains a quadratic equation for the (complex) amplification factor, G

$$(1 + K_x + K_y)\,G^2 - 2(K_x \cos\alpha + K_y \cos\beta - iA \sin\alpha - iB \sin\beta)\,G$$
$$- (1 - K_x - K_y) = 0, \tag{1}$$

where $K_x = 2\kappa_x\,\Delta t/\Delta x^2$, $K_y = 2\kappa_y\,\Delta t/\Delta y^2$, $A = U\,\Delta t/\Delta x$, $B = V\,\Delta t/\Delta y$, $\alpha = m\,\Delta x$, and $\beta = l\,\Delta y$. The angles α and β vary between $-\pi$ and π depending on the wavenumbers m and l.

The von Neumann necessary stability criterion states that the scheme is stable if $|G| \leqslant 1$ for all α and β, and unstable otherwise. This leads to an inequality involving K_x, K_y, A, and B, hence U, V, κ_x, κ_y, Δx, Δy, Δt. Since the solution of (1) involves taking the square root of a complex number, the problem immediately leads to mathematical difficulties. An alternative method of solution is followed. The solution to the above problem is greatly facilitated by the construction of a positive-definite function.

DEFINITION.

$$E(\alpha, \beta) \equiv \left(\frac{K_x}{S}\cos\alpha + \frac{K_y}{S}\cos\beta\right)^2 + (A \sin\alpha + B \sin\beta)^2, \text{ where } S = K_x + K_y. \tag{2}$$

THEOREM I. *The numerical scheme is stable if $E \leqslant 1$ for all α and β, and unstable otherwise. In other words, the expression E conveniently replaces the more complicated expression $|G|$ in the stability discussion.*

Proof. The complex variable G is written as $R \exp i\theta$, R being its modulus. Equation (1) is then split into its real and imaginary components. If $X = (K_x \cos\alpha + K_y \cos\beta)/S$ and $Y = A \sin\alpha + B \sin\beta$, these are

$$(1 + S)\,R^2 \cos 2\theta - 2SXR \cos\theta - 2YR \sin\theta - (1 - S) = 0,$$
$$(1 + S)\,R^2 \sin 2\theta - 2SXR \sin\theta + 2YR \cos\theta = 0.$$

Solving for X and Y and forming $E = X^2 + Y^2$, one obtains

$$E = \frac{(R^2+1)^2}{4R^2} + \frac{R^4 - 1}{2R^2}\left(\frac{\cos^2\theta}{S} + S\sin^2\theta\right) + \frac{(R^2-1)^2}{4R^2}\left(\frac{\cos^2\theta}{S^2} + S^2\sin^2\theta\right).$$

Two cases are possible, either $S \leqslant 1$ or $S \geqslant 1$. One can concentrate only on the case $S \leqslant 1$, since the other case is obtained by substitution of S by S^{-1} and θ by $\pi/2 - \theta$. For $S \leqslant 1$, the extreme values of E over the range of θ are

$$E_{\max} = \left[\frac{(R^2+1) + (R^2-1)/S}{2R}\right]^2,$$

$$E_{\min} = \left[\frac{(R^2+1) + S(R^2-1)}{2R}\right]^2.$$

One observes that for $R = (1-S)/(1+S)$ and $R = 1$, $E_{\max} = E_{\min} = 1$, that both $E_{\max}$ and $E_{\min}$ are less than one for $(1-S)/(1+S) < R < 1$, and that both $E_{\max}$ and $E_{\min}$ are greater than one outside of that range.

For $S \geqslant 1$, the results are similar except for S being replaced by S^{-1}. Both cases can be summarized as follows (see Fig. 1): $E = 1$ for $R = |1-S|/|1+S|$ and $R = 1$, $E < 1$ for R between these values, and $E > 1$ for R outside that range.

On the other side, the ratio of the last term to the first term of Eq. (1) implies that the product of the two roots, G_1 and G_2, of that equation equals $-(1-S)/(1+S)$, i.e.,

$$|G_1| \cdot |G_2| = \left|\frac{1-S}{1+S}\right|.$$

This implies that, when the modulus of one root is less that $|1-S|/|1+S|$, the modulus of the other ought to be greater than unity, and the scheme is numerically unstable. Since R is the modulus of either root, one concludes that the regions of stability and instability correspond exactly to the regions where E is less than or greater than unity, respectively. Theorem I is thus proven.

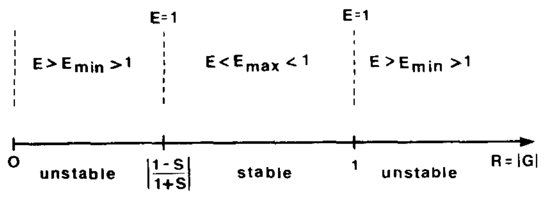

FIG. 1. Summary of results of Theorem I.

THEOREM II. *The criterion*

$$\frac{A^2}{K_x}+\frac{B^2}{K_y}\leqslant\frac{1}{S}, \tag{3}$$

is the necessary and sufficient condition under which E is less than or equal to unity. In conjunction with Theorem I, this theorem establishes that (3) is the stability criterion of the numerical scheme.

Proof. The extrema of E over the range of values of α and β are reached for $\partial E/\partial\alpha=\partial E/\partial\beta=0$, i.e.,

$$X\frac{K_x}{S}\sin\alpha-YA\cos\alpha=0,$$

$$X\frac{K_y}{S}\sin\beta-YB\cos\beta=0,$$

where X and Y are functions of α and β defined above. Considering these equations as a 2×2 linear system for the unknowns X and Y, one concludes that, if the determinant of that system is nonzero, $X=Y=0$. This solution (if it exists) corresponds to the minimum of the positive-definite expression $E=X^2+Y^2$. The maxima, and possible other minima, of E are thus obtained when the system determinant vanishes

$$\frac{K_x}{A}\tan\alpha=\frac{K_y}{B}\tan\beta.$$

This constraint between α and β corresponds to a curve in the (α,β) plane which can be described parametrically as

$$\tan\alpha=\frac{A}{K_x}\lambda,\qquad \tan\beta=\frac{B}{K_y}\lambda,$$

where the parameter λ can vary from $-\infty$ to $+\infty$. Along this curve, the expression E takes the form of a function of λ^2

$$E(\lambda^2)=\frac{1}{S^2}\left[\frac{K_x}{(1+A^2\lambda^2/K_x^2)^{1/2}}+\frac{K_y}{(1+B^2\lambda^2/K_y^2)^{1/2}}\right]^2$$
$$+\lambda^2\left[\frac{A^2/K_x}{(1+A^2\lambda^2/K_x^2)^{1/2}}+\frac{B^2/K_y}{(1+B^2\lambda^2/K_y^2)^{1/2}}\right]^2.$$

End values are $E(0)=1$ and $E(\infty)=(|A|+|B|)^2$. Extrema of E are found for $dE/d\lambda^2=0$. This occurs when

$$\left(\frac{A^2}{K_x}-\frac{K_x}{S^2}\right)\left(1+\frac{A^2}{K_x^2}\lambda^2\right)^{-1/2}+\left(\frac{B^2}{K_y}-\frac{K_y}{S^2}\right)\left(1+\frac{B^2}{K_y^2}\lambda^2\right)^{-1/2}=0.$$

This equation can be rewritten in the form $a\lambda^2 + b = 0$ and thus has at most one positive root for λ^2. This implies that $E(\lambda^2)$ either varies monotonically from 1 to $(|A| + |B|)^2$ or has one extremum between these values.

The Taylor expansion of $E(\lambda^2)$ about $\lambda^2 = 0$ is

$$E(\lambda^2) = 1 + \left(\frac{A^2}{K_x} + \frac{B^2}{K_y}\right)\left(\frac{A^2}{K_x} + \frac{B^2}{K_y} - \frac{1}{S}\right)\lambda^2 + O(\lambda^4).$$

Two cases are possible. Either $A^2/K_x + B^2/K_y - 1/S$ is greater than zero or not. In the former case, E is greater than unity in the vicinity of $\lambda^2 = 0$ (i.e., in the vicinity of $\alpha = \beta = 0$ or $\pm\pi$), and thus inequality (3) is a necessary condition to ensure $E \leqslant 1$. In the latter case, E decreases and is less than unity in the vicinity of $\lambda^2 = 0$. By virtue of the previous result, E either decreases monotonically or reaches a minimum. Therefore, to determine whether E can be greater than one, one ought to compare its value for $\lambda^2 = \infty$ to unity. This value can be written as

$$\begin{aligned}
E(\infty) &= (|A| + |B|)^2 \\
&= (K_x + K_y)\left(\frac{A^2}{K_x} + \frac{B^2}{K_y}\right) - \left(\frac{K_y}{K_x}A^2 + \frac{K_x}{K_y}B^2 - 2|A||B|\right) \\
&= 1 + S\left(\frac{A^2}{K_x} + \frac{B^2}{K_y} - \frac{1}{S}\right) - \left[\left(\frac{K_y}{K_x}\right)^{1/2}|A| - \left(\frac{K_x}{K_y}\right)^{1/2}|B|\right]^2 \\
&\leqslant 1 + S\left(\frac{A^2}{K_x} + \frac{B^2}{K_y} - \frac{1}{S}\right),
\end{aligned}$$

and is therefore less than one when

$$\frac{A^2}{K_x} + \frac{B^2}{K_y} \leqslant \frac{1}{S},$$

i.e., when (3) is met. Therefore, inequality (3) is also a sufficient condition to ensure $E \leqslant 1$.

Collecting all the above results, one concludes that E is always less than or equal to one for all values of α and β provided that (3) is met, and that there exist values of α and β in the vicinity of $\alpha = \beta = 0$ and $\pm\pi$ for which E is greater than one provided that (3) is not met. Hence, Theorem II is proven.

3. Discussion

Theorems I and II establish that (3) is the stability requirement of the combined leap-frog Dufort–Frankel scheme. With the original notation, this criterion imposes

$$\left(\frac{\kappa_x}{\Delta x^2} + \frac{\kappa_y}{\Delta y^2}\right)\left(\frac{U^2}{\kappa_x} + \frac{V^2}{\kappa_y}\right)\Delta t^2 \leqslant 1. \tag{4}$$

Generalized to three dimensions, the criterion is

$$\left(\frac{\kappa_x}{\Delta x^2}+\frac{\kappa_y}{\Delta y^2}+\frac{\kappa_z}{\Delta z^2}\right)\left(\frac{U^2}{\kappa_x}+\frac{V^2}{\kappa_y}+\frac{W^2}{\kappa_z}\right)\Delta t^2 \leqslant 1.$$

It is important to note that this inequality does not depend explicitly on the magnitude of the diffusivity coefficients but only on their ratio. If both diffusivities are equal, the criterion becomes

$$\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)(U^2+V^2)\,\Delta t^2 \leqslant 1.$$

However, the influence of the diffusive terms in the problem is not absent, for the stability criterion of the purely advective problem is different. Indeed, for $\kappa_x = \kappa_y = 0$, Eq. (1) governing the amplification factor becomes

$$G^2 + 2i(A\sin\alpha + B\sin\beta)\,G - 1 = 0,$$

and the moduli of its roots are not greater than one if and only if $(A\sin\alpha + B\sin\beta)^2 \leqslant 1$ for all α and β. Thus, $|A| + |B| \leqslant 1$ or

$$\left(\frac{|U|}{\Delta x}+\frac{|V|}{\Delta y}\right)\Delta t \leqslant 1. \tag{5}$$

Criterion (5) is the stability requirement for the leap-frog scheme alone. It is the criterion upon which users rely who follow the rule of thumb (meet the requirements of every scheme separately). The correct criterion in combination with a Dufort–Frankel scheme, i.e., (4), is somewhat more restrictive. Indeed,

$$\begin{aligned}\left(\frac{\kappa_x}{\Delta x^2}+\frac{\kappa_y}{\Delta y^2}\right)\left(\frac{U^2}{\kappa_x}+\frac{V^2}{\kappa_y}\right)\Delta t^2 &= \left(\frac{|U|}{\Delta x}+\frac{|V|}{\Delta y}\right)^2\Delta t^2 + \left[\left(\frac{\kappa_y}{\kappa_x}\right)^{1/2}\frac{|U|}{\Delta y}-\left(\frac{\kappa_x}{\kappa_y}\right)^{1/2}\frac{|V|}{\Delta x}\right]^2\Delta t^2\\ &\geqslant \left(\frac{|U|}{\Delta x}+\frac{|V|}{\Delta y}\right)^2\Delta t^2.\end{aligned}$$

The difference is particularly significant if one chooses $\kappa_x = \kappa_y$ and a small grid size in the direction perpendicular to the direction of the largest velocity component. This situation may arise in the study of geophysical fronts where the velocity is almost perpendicular to the direction of maximum gradients. In such case, one has to compromise between either a small grid size in the other direction or a small time step.

The stability loss resulting from this added constraint can also be illustrated geometrically in a (A, B) plane (Fig. 2). In this plane, the criterion obtained in the

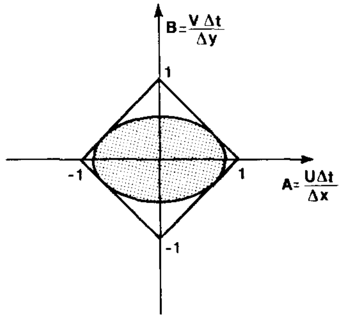

FIG. 2. Geometrical interpretation of the criteria with and without diffusive terms (ellipse and square, respectively). Regions of stability lie inside these curves. Since the ellipse is situated inside the square, the stability criterion is more restrictive in the presence of diffusion than in its absence.

absence of diffusivity, $|A| + |B| \leqslant 1$, is represented by a square of side $2^{1/2}$ and oriented at 45° with respect to the coordinate axes. The region of stability lies inside the square. In the presence of diffusivity, the criterion is (3) and is represented by an ellipse lying inside the square and tangent to its four sides. The eccentricity of the ellipse varies with the ratio K_x/K_y, i.e., with κ_x/κ_y and $\Delta x/\Delta y$. The region of stability lies inside the ellipse. Since the ellipse is always situated inside the square, criterion (3) is the most stringent of the two. The area outside the ellipse but inside the square represents the stability loss due to the presence of diffusivity, although the magnitude of the diffusivity coefficient does not intervene. This area is minimum when the ellipse becomes a circle ($\kappa_x/\Delta x^2 = \kappa_y/\Delta y^2$). From this observation stems a recommendation: to minimize the stability restriction resulting from the combination of leap-frog and Dufort–Frankel schemes, it is best to choose grid sizes of both directions in the ratio of the square roots of the diffusivity coefficients.

Finally, it should be noted that, in one dimension, criteria (4) and (5) are identical. This explains why the one-dimensional studies of the combined schemes [1, 4] do not mention the possibility of additional constraints resulting from the combination of schemes.

4. Generalization of the Method of Solution

This section presents the extension of the above methodology to more general cases for which the amplification factor G is given by the quadratic equation

$$(a + ia')\,G^2 + (b + ib')\,G + (c + ic') = 0, \tag{6}$$

where a, a', b, b', c, and c' are all real coefficients depending on some angles $(\alpha, \beta, \ldots)$. The problem is to determine the structure of an expression E, formed with these coefficients, which is less than or equal to one when the moduli of both roots for G are less than or equal to one, and is greater than one otherwise. In other words, one is seeking a function which possesses the same properties as $|G|$ and, yet, has a simpler expression. Such function is constructed in the first part of this section, while the second part is devoted to the establishment of its properties (Theorem III). Finally, the third part treats particular cases.

Construction of the Adequate Function

In view of its desired properties, the required expression must, at least, be equal to unity when the modulus of the amplification factor, $|G|$, is also unity. Hence, one is seeking a mathematical constraint between the coefficients of (6) which must hold when $|G|$ is one. Equation (6) can be written as

$$(a + ia')\, G + b + ib' + (c + ic')\, G^{-1} = 0, \tag{7}$$

and, with $G = \exp(i\theta) = \cos\theta + i\sin\theta$, $G^{-1} = \exp(-i\theta) = \cos\theta - i\sin\theta$, can be split into its real and imaginary components. These are

$$(a + c)\cos\theta - (a' - c')\sin\theta = -b,$$
$$(a' + c')\cos\theta + (a - c)\sin\theta = -b'.$$

This linear system for the unknowns $\cos\theta$ and $\sin\theta$ has a unique solution when its determinant

$$\Delta = A^2 - C^2 = (A + C)(A - C),$$

is nonzero. In what follows, A and C denote the moduli of $a + ia'$ and $c + ic'$, respectively. The particular case $A = C$ ($\Delta = 0$) is discussed at the end of the section. As long as A differs from C, one can solve for $\cos\theta$ and $\sin\theta$, and then form the combination $\cos^2\theta + \sin^2\theta$ to eliminate θ. The result is

$$1 = \frac{1}{\Delta^2}\,[b(a - c) + b'(a' - c')]^2 + \frac{1}{\Delta^2}\,[b(a' + c') - b'(a + c)]^2.$$

A potential candidate for the desired expression which must be equal to one when $|G|$ is unity is thus

$$E \equiv \frac{1}{\Delta^2}\,[b(a - c) + b'(a' - c')]^2 + \frac{1}{\Delta^2}\,[b(a' + c') - b'(a + c)]^2, \tag{8}$$

where $\Delta = (a^2 + a'^2) - (c^2 + c'^2) = A^2 - C^2$. If $a' = c' = 0$, this expression becomes

$$E = \left(\frac{b}{a + c}\right)^2 + \left(\frac{b'}{a - c}\right)^2.$$

Miller [2] devised an algorithm to derive the necessary and sufficient conditions under which all roots of any given polynomial fall inside or on the unit circle in the complex plane. His algorithm is based on a systematic reduction to polynomials of lesser degrees. From any polynomial of degree n

$$f(G) = \sum_{j=0}^{n} a_j G^j,$$

one can form a reduced polynomial of degree $n-1$

$$f_1(G) = \frac{1}{G} \sum_{j=0}^{n} [\bar{a}_n a_j - a_0 \bar{a}_{n-j}]\, G^j.$$

If $f(G)$ is now quadratic as in (7), then $f_1(G)$ is linear

$$f_1(G) = (A^2 - C^2)\, G + (a - ia')(b + ib') - (c + ic')(b - ib'),$$

and its only root is easily determined. Expression (8) is then found to be nothing but the modulus of that root.

THEOREM III (A Generalization of Theorem I). *For $C < A$, the expression E defined by* (8) *is less than one when both roots of* (6) *have moduli less than one, is equal to one when either root of* (6) *has a modulus equal to one, and is greater than one if one of the roots of* (6) *has a modulus greater than one. In other words, the numerical scheme leading to* (6) *is stable as long as $C < A$ and $E \leqslant 1$ for all values of the variable angles.* (*The cases $C = A$ and $C > A$ are treated separately afterward.*)

Proof. If $|G| = R$, $G = R\cos\theta + iR\sin\theta$, $G^{-1} = R^{-1}\cos\theta - iR^{-1}\sin\theta$, the real and imaginary parts of (7) become

$$b = -\left(aR + \frac{c}{R}\right)\cos\theta + \left(a'R - \frac{c'}{R}\right)\sin\theta,$$

$$b' = -\left(a'R + \frac{c'}{R}\right)\cos\theta - \left(aR - \frac{c}{R}\right)\sin\theta.$$

Replacing b and b' by the above expressions into the definition (8) of E, one obtains

$$E = \frac{1}{\Delta^2}\left(A^2R - \frac{C^2}{R}\right)^2 + \frac{A^2C^2}{\Delta^2}\left(R - \frac{1}{R}\right)^2$$
$$+ \frac{2}{\Delta^2}\left(A^2R - \frac{C^2}{R}\right)\left(R - \frac{1}{R}\right)[(a'c - ac')\sin 2\theta - (ac + a'c')\cos 2\theta]. \quad (9)$$

One verifies immediately that the above expression reaches the value one when $R = 1$ and $R = C/A$, i.e., when the modulus of either root of (6) is equal to unity. The values of E for other values of R are discussed based on the extremal values of E with respect to θ. These extrema are reached for

$$\cos 2\theta = \varepsilon \frac{ac + a'c'}{AC}, \qquad \sin 2\theta = -\varepsilon \frac{a'c - ac'}{AC}, \qquad \varepsilon = \pm 1,$$

and can be expressed as

$$E^* = 1 + \frac{(R^2 - 1)(A^2 R^2 - C^2)}{(A + \varepsilon C)^2 R^2}.$$

From this expression, it is evident that $E^* \leqslant 1$ for $C/A \leqslant R \leqslant 1$ and $E^* > 1$ otherwise, regardless of the value of ε (+1 or −1). This implies that inside the interval $[C/A, 1]$, the maximum value of E and hence all values of E are less than or equal to unity, while outside that interval, the minimum value of E and hence all values of E are greater than one.

Since the two roots of (6) are such that the product of their moduli is equal to C/A and less than one, both moduli are less than one if either one is less than one and greater than C/A (i.e., the other less than one). This final remark in conjunction with the previous result establishes the theorem.

Particular Cases

In the case $A = C$, the two roots of (6) have a product whose modulus equals one. Hence, both moduli are no greater than unity if and only if they are each equal to one. When this is the case, Eq. (6) can be written as

$$Ae^{i(\alpha + \theta)} + Be^{i\beta} + Ae^{i(\gamma - \theta)} = 0,$$

where $a + ia' = A \exp(i\alpha)$, $b + ib' = B \exp(i\beta)$, $c + ic' = A \exp(i\gamma)$, and $G = \exp(i\theta)$. It can also be expressed as

$$-\frac{B}{A} = e^{i(\alpha + \theta - \beta)} + e^{i(\gamma - \theta - \beta)},$$

i.e., the real, negative number $-B/A$ must be equal to the sum of two complex numbers of modulus one. This implies

$$2\beta - \alpha - \gamma = 0 \qquad \text{or} \qquad 2\pi,$$

and

$$B \leqslant 2A.$$

Therefore, unless these two conditions are met for all values of the variable parameters of Eq. (6), the scheme is unstable.

Finally, in the case $C > A$, there is always at least one root of (6) which has a modulus greater than one. And thus, if there exists at least one combination of parameters in Eq. (6) which leads to $C > A$, the scheme that led to that equation is unconditionally unstable.

The above theorem is a particular case of a much more general theorem stated in [2], where Miller establishes that a polynomial, $f(G)$, of any degree has all its roots inside or on the unit circle if and only if either (i) $|a_0| < |a_n|$ and $f_1(G)$ has all its roots inside or on the unit circle, or (ii) $f_1(G) \equiv 0$ and $f'(G)$ has all its roots inside or on the unit circle. The above proof of Theorem III demonstrates proposition (i) in the case when $f(G)$ is quadratic since the modulus of the only root of $f_1(G)$ is the expression E, while the treatment of the particular case $A = C$ covers proposition (ii).

The majority of numerical schemes leads to quadratic equations for the amplification factor, G. Therefore, although Miller's theorem is very general, it was felt necessary to state explicitly which criteria have to be met in the particular quadratic case, and to provide a straightforward proof for this particular case.

5. Conclusions

The mathematical difficulty inherent in the stability study of the mixed leap-frog Dufort–Frankel numerical scheme for advective–diffusive problems is overcome by the definition of a certain positive-definite expression, not equal to, but, conveniently, replacing the modulus of the amplification factor. The results demonstrate that, in two dimensions, the scheme must obey a condition more restrictive than the one which would result from the sole leap-frog scheme ((4) instead of (5)). Although the stability loss results from the presence of diffusive terms, the condition does not depend explicitly on the magnitude of the diffusivity coefficients, but only on their ratio. To minimize the stability restriction, it is recommended to choose grid spacings in both directions in the same ratio as the square roots of the diffusivity coefficients. It is also noted that, in one dimension, the presence of diffusion does not affect the stability criterion resulting from a leap-frog scheme alone.

Finally, for all cases when the amplification factor can be given by a complex, quadratic equation, a general expression of the positive-definite expression is derived and its properties are studied. It is also shown how the same expression results from Miller's algorithm [2]. This extension of the methodology should facilitate the study of various combined schemes whose stability has not yet been rigorously established.

Acknowledgments

The author wishes to thank Dr. J. J. O'Brien for suggesting this problem and for his encouragements during the research, and an anonymous reviewer for providing the reference to Miller's work. This research was supported by the Office of Naval Research and is Contribution 194 of the Geophysical Fluid Dynamics Institute.

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